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Equilibrium Points of Nonatomic Games[1]

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The Nash theorem on the existence of equilibrium points in N-person noncooperative games in normal form is generalized to the case when there is a continuum of players endowed with a nonatomic measure. The mathematical tools are those used in mathematical economics, in particular, markets with a continuum of traders. The main result shows that under a restriction on the payoff functions there exists an equilibrium in pure strategies.

KEY WORDS: Games; nonatomic games; equilibrium point, set-valued function; fixed point.

A nonatomic game is a game where the set of the players is endowed with a nonatomic measure. The purpose of this paper is to define a nonatomic game in a normal form, to define its equilibrium points in the sense of Nash,[(4)] and to prove their existence.

Nonatomic games enable us to analyze a conflict situation where the single player has no influence on the situation but the agregative behavior of "large" sets of players can change the payoffs. The examples are numerous: Elections, many small buyers from a few competing firms, drivers that can choose among several roads, and so on.

In our model the set of players T is the unit interval [0, 1] endowed with Lebesgue measure λ. Each player has to choose one of n activities. We represent an activity (or a pure strategy of a player) by a basis vector e_i in R^n,

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the n-dimensional Euclidean space. The vector e_i is the vector in R^n with one as ith coordinate and zero otherwise. Denote

$$P = \left\{ x = (x^1,..., x^n) \in R^n \mid x^i \geqslant 0,\ 1 \leqslant i \leqslant n,\ \sum_{i=1}^{n} x^i = 1 \right\}$$

Then $P = \text{conv}(\{e_1 ,..., e_n\})$ (conv stands for convex hull) and it is the set of mixed strategies of each player. A *T-strategy* is a measurable function $\hat{x}$ from T to P. Thus, $\hat{x} = (\hat{x}^1,..., \hat{x}^n)$, where $\hat{x}^i$ is a measurable real-valued function from T to $[0, 1]$. In this case $\hat{x}^i$ is Lebesgue-integrable and we write $\int_S \hat{x}$ for $(\int_S \hat{x}^1(t)\, d\lambda,..., \int_S \hat{x}^n(t)\, d\lambda)$. As usual we neglect the distinction between integrable functions and equivalence classes of such functions. Hence, a T-strategy $\hat{x}$ belongs to $L_1(T \times \{1,..., n\})$. Let $\hat{P}$ denote the set of all T-strategies endowed with L_1 weak topology. The set $\hat{P}$ is a compact, convex subset of a locally convex linear topological space.

Before defining the payoff function we define an auxiliary (utility) function, $\hat{u}(\cdot, \cdot) : T \times \hat{P} \to R^n$. Now, $\hat{u}^i(t_0 , \hat{x})$ describes the utility of player t_0 when a.e. player chooses $\hat{x}(t)$ and t_0 chooses e_i. So the payoff to player t, h_t, is defined as

$$h_t(\hat{x}) = \hat{x}(t) \cdot \hat{u}(t, \hat{x})$$

where $x \cdot y$ denotes the inner product in R^n for $x, y \in R^n$.

Thus a nonatomic game in a normal form is defined completely by the function $\hat{u}$.

We shall need the following conditions:

(a) For all t in T, $\hat{u}(t, \cdot)$ is continuous on $\hat{P}$.

(b) For all $\hat{x}$ in $\hat{P}$ and $i, j = 1,..., n$, the set $\{t \in T \mid \hat{u}^i(t, \hat{x}) > \hat{u}^j(t, \hat{x})\}$ is measurable.

A T-strategy $\hat{x}$ is *in equilibrium* iff, a.e.,

$$\forall p \in P \qquad h_t(\hat{x}) \geqslant p \cdot \hat{u}(t, \hat{x})$$

Theorem 1. A nonatomic game in a normal form fulfilling conditions (a) and (b) has a T-strategy in equilibrium.

A T-strategy $\hat{x}$ is called pure iff, a.e., $\hat{x}(t) \in \{e_1 ,..., e_n\}$, i.e., almost each player uses a pure strategy. Our main result is the following theorem.

Theorem 2. If in addition to the conditions of Theorem 1, a.e., $\hat{u}(t, \hat{x})$ depends only on $\int_T \hat{x}$, then there is a pure T-strategy in equilibrium.

The importance of Theorem 2 lies in the fact that in many real, gamelike situations, a mixed strategy has no meaning. The additional condition is not too restrictive, as is explained later in remark 2.

A game in a normal form is finite if there is a finite number of players and each player has a finite number of pure strategies.

Corollary (Nash theorem): Every finite game has a strategy in equilibrium.

We shall show that this is a simple corollary of Theorem 2 (and not of Theorem 1).

Proof of Theorem 1. For player t and T-strategy $\hat{x}$, set

$$B(t, \hat{x}) = \{p \in P \mid \forall q \in P : p \cdot \hat{u}(t, \hat{x}) \geqslant q \cdot \hat{u}(t, \hat{x})\}$$

This is the set of the best answers for player t when T-strategy $\hat{x}$ is chosen. Obviously $B(t, \hat{x})$ is convex and nonempty.

Claim 1. For each t the graph of $B(t, \cdot)$ is closed in $\hat{P} \times P$.

Given t, let $\hat{x}_n \to \hat{x}_0$, $p_n \to p_0$ and for each q in P,

$$p_n \cdot \hat{u}(t, \hat{x}_n) \geqslant q \cdot \hat{u}(t, \hat{x}(t, \hat{x}_n), \qquad n = 1, 2, \ldots$$

Because of the continuity of $\hat{u}(t, \cdot)$, the inequality holds in the limit. So, the proof of claim 1 is completed.

We define a set-valued function $\alpha : \hat{P} \to \hat{P}$ by

$$\alpha(\hat{x}) = \{\hat{y} \in \hat{P} \mid \text{a.e. } \hat{y}(t) \in B(t, \hat{x})\}$$

Claim 2. For each $\hat{x}$, $\alpha(\hat{x})$ is nonempty and convex.

Define for $i = 1, \ldots, n$

$$T_i = \{t \in T \mid \hat{u}^j(t, \hat{x}) \leqslant \hat{u}^i(t, \hat{x}), \qquad j = 1, \ldots, n\}$$

One has $\bigcup_{i=1}^{n} T_i = T$ and for $t \in T_i$, $e_i \in B(t, \hat{x})$. Because of condition (b), T_i is measurable. Let $S_1 = T_1$ and $S_i = T_i \setminus (\bigcup_{j=1}^{i=1} T_j)$, $i = 2, \ldots, n$. The T-strategy $\hat{y}$, defined by $\hat{y}(t) = e_i$ for $t \in S_i$, belongs to $\alpha(\hat{x})$. The convexity of $B(t, \hat{x})$ implies that of $\alpha(\hat{x})$.

Claim 3. The graph of α is closed in $\hat{P} \times \hat{P}$.

Let $\hat{x}_n \to \hat{x}_0$, $\hat{y}_n \to \hat{y}_0$ and for $n \geqslant 1$, $\hat{y}_n \in \alpha(\hat{x}_n)$. Assume per absurdum that for a nonnull, measurable subset S of T, $\hat{y}_0(t) \notin B(t, \hat{x}_0)$. For each t, $B(t, \hat{x}_0)$ is a convex hull of a subset of the set $\{e_1, \ldots, e_n\}$. So there is a nonnull, measurable subset V of S and a subset $\{e_{i_1}, \ldots, e_{i_k}\}$ of $\{e_1, \ldots, e_n\}$ such that for each $t \in V$, $B(t, \hat{x}_0) = \text{conv}(\{e_{i_1}, \ldots, e_{i_k}\})$ and $\hat{y}_0(t) \notin B(t, \hat{x}_0)$. Hence, there is p in P such that $p \cdot \hat{y}_0(t) > 0$ and $p \cdot e_{i_j} = 0$, $j = 1, \ldots, k$. So $p \cdot \int_V \hat{y}_0 > 0$, but for each $\hat{y}$ in $\hat{P}$ with $\hat{y}(t) \in B(t, \hat{x}_0)$ for $t \in V$, $p \cdot \int_V \hat{y} = 0$. Now, $\hat{y}_n \to \hat{y}_0$ implies $\int_V \hat{y}_0 = \lim_n \int_V \hat{y}_n$. Using Aumann's[1] Proposition 4.1, we get $\lim_n \int_V \hat{y}_n \subset \int_V \lim_n \operatorname{Sup}\{\hat{y}_n(t)\}$, where

$$\int_V \lim_n \operatorname{Sup}\{\hat{y}_n(t)\} = \left\{\int_V \hat{y} \mid \text{a.e. in } V, \hat{y}(t) \text{ is a limit point of } \{\hat{y}_n(t)\}_{n=1}^{\infty}\right\}.$$

But because of claim 1, each limit point of $\{\hat{y}_n(t)\}_{n=1}^{\infty}$ belongs to $B(t, \hat{x}_0)$—a contradiction.

To complete the proof of Theorem 1, notice that by claims 2 and 3, α fulfills the conditions of the Fan–Glicksberg[2,3] fixed-point theorem and if $\hat{x} \in \alpha(\hat{x})$, then obviously $\hat{x}$ is an equilibrium.

Proof of Theorem 2. By Theorem 1, there is a T-strategy $\hat{x}$ in equilibrium. We have to prove that there is a pure T-strategy $\hat{y}$ such that $\int_T \hat{x} = \int_T \hat{y}$ and, a.e., $\hat{y}(t) \in B(t, \hat{x})$. As was mentioned previously,

$$B(t, \hat{x}) = \operatorname{conv}(\{e_i \mid e_i \in B(t, \hat{x})\}).$$

From Aumann's[1] Theorem 3, we have

$$(c) \qquad \int_T B(t, \hat{x}) = \int_T \{e_i \mid e_i \in B(t, \hat{x})\}$$

if $\{(t, e_i) \mid e_i \in B(t, \hat{x})\}$ is a Borel subset of $T \times R^n$, where

$$\int_T B(t, \hat{x}) = \left\{\int_T \hat{y} \mid \hat{y} \in \hat{P} \text{ and a.e. } \hat{y}(t) \in B(t, \hat{x})\right\}$$

$$\int_T \{e_i \mid e_i \in B(t, \hat{x})\} = \left\{\int_T \hat{y} \mid \hat{y} \in \hat{P} \text{ and a.e. } \hat{y}(t) \in \{e_i \mid e_i \in B(t, \hat{x})\}\right\}$$

Of course, $\int_T \hat{x}$ belongs to the left side of (c). To complete the proof, we shall demonstrate the Borel measurability condition. The set-valued function $\{e_i \mid e_i \in B(\cdot, \hat{x})\}$ attains a finite number of values, and each of them is a finite subset of R^n and hence a Borel set in R^n. Given a subset $\{e_{i_1}, \dots, e_{i_k}\}$ of $\{e_1, \dots, e_n\}$ we have to show that $\{t \mid \operatorname{conv}(\{e_{i_1}, \dots, e_{i_k}\} = B(t, \hat{x})\}$ is Borel subset of T. The last statement is implied by condition (b).

Proof of the Corollary. Let m be the number of players in the finite game. For $i = 1, \dots, m$ let K_j be the set of pure strategies of player j. We assume that $|K_j|$, the cardinality of K_j, is finite. We assume also, without loss of generality, that the payoff function of player j has nonnegative values. We represent this finite game by a nonatomic game where $n = \sum_{j=1}^{m} |K_j|$. Let

$(T_j)_{j=1}^m$ be a partition of T with $\lambda(T_j) = 1/m$ for every j. Given t in T_j and a pure T-strategy $\hat{x}$, we have to define $\hat{u}^i(t, \hat{x})$. If $i \notin K_j$, we define $\hat{u}^i(t, \hat{x}) = -1$. If $i \in K_j$, we define $\hat{u}^i(t, \hat{x})$ as the payoff obtained by player j in the finite game when he plays the pure strategy i and player $j'(j' \neq j)$ plays $i' \in K_{j'}$ with probability $m\lambda(\{t \in T_{j'}\ \hat{x}(t) = i'\})$. We choose an arbitrary k in $K_{j'}$ and attach to it the probability $m\lambda(\{t \in T_{j'} \mid \hat{x}(t) = k \text{ or } \hat{x}(t) \notin K_{j'}\})$. Thus $\hat{u}$ is well defined and depends on $\int_T \hat{x}$ only. The definition of $\hat{u}$ implies also that if $\hat{x}$ is a pure T-strategy in equilibrium, then, a.e. in T_j, $\hat{x}(t) \in K_j$, for every j. Hence, a pure T-strategy, which exists by Theorem 2, induces a strategy in equilibrium in the finite game.

Remarks

1. The dimension of R^n means that the number of activities of all players is uniformly bounded on T and it does not mean that all the players have the same number of different choices.

2. Theorem 2 can be generalized in the following manner: Instead of "$\hat{u}(t, \hat{x})$ depends only on $\int_T \hat{x}$" once can assume "$\hat{u}(t, \hat{x})$ depends only on $\{\int_{T_i} \hat{x}\}_{i=1}^k$", where $\{T_i\}_{i=1}^k$ are Lebesgue-measurable subsets of T, $k = 1, 2, \ldots$. A similar proof applies (because the last restriction is equivalent to "$\{T_i\}_{i=1}^k$ is a measurable partition of T.")

3. The set of all pure T-strategies is dense in $\hat{P}$ (in the weak topology). Hence the function $\hat{u}(t, \cdot)$, which is continuous on $\hat{P}$, is determined by its values on the pure T-strategies. So the following problem is suggested: "Is the conclusion of Theorem 2 true under the conditions of Theorem 1?" We shall answer negatively by the following example: Let $n = 2$ and for $i = 1, 2$ we define $u^i(t, \hat{x}) = \| te_i - \int_{[0,t)} \hat{x} \|$, where $\| \cdot \|$ denotes the distance in R^n. Assume that $\hat{x}$ is in equilibrium and that, a.e., $\hat{x}(t) \in \{e_1, e_2\}$. First, we show that the set $S = \{t \mid \int_{[0,t)} \hat{x} = \frac{1}{2}t(e_1 + e_2)\}$ is null. Otherwise there is an interval $[r, s] \subset S$ with $0 < r < s < 1$. This, in turn, implies that the density of the set $E_i = \{t \in [r, s] | \ \hat{x}(t) = e_i\}$ in the interval $[r, s]$ is $\frac{1}{2}$ for $i = 1, 2$, which is impossible. [There is no Lebesgue-measurable set E on the real line such that for every Lebesgue-measurable set F in $[r, s]$, $\lambda(E \cap F) = \frac{1}{2}\lambda(F)$.]

Let $0 < s \leqslant 1$ be such that $\| se_2 - \int_{[0,s)} \hat{x} \| < \| se_1 - \int_{[0,s)} \hat{x} \|$. Because of the continuity in t of $\int_{[0,t)} \hat{x}$ there is a first r such that $r < s$ and for all $r < t \leqslant s$ the inequality $\| te_2 - \int_{[0,t)} \hat{x} \| < \| te_1 - \int_{[0,t)} \hat{x} \|$ holds. Hence for a.e. $t \in [r, s]$ the assumption that $\hat{x}$ is in equilibrium implies that $\hat{x}(t) = e_1$; a contradiction to the inequality in s. QED.

4. Although the main result of this work is Theorem 2, there is some interest in Theorem 1. One may ask, and the referee did, whether the techniques of the proof of Theorem 1[(1)] could yield a stronger result. The

answer is positive. The generalization of Theorem 1 is obtained not by weakening one of the explicit assumptions (a) or (b) but by weakening the assumption that P is a simplex (i.e., a convex hull of linearly independent vectors in R^n) and that $\hat{u}(t, \cdot)$ is affine on P. Instead we assume that for each t in T the set P_t is a compact, convex subset of R^n. A T-strategy is a measurable function $\hat{x}$ from T to R^n, s.t., $\hat{x}(t) \in P_t$ for all t. Now, $\hat{P}$ denotes the set of all T-strategies, and set $G = \{(t, x) \in T \times R^n \mid x \in P_t\}$.

Using Aumann's[1] Theorem 2, $\hat{P}$ is nonempty if we assume the following condition: (A) G is a Borel subset of R^{n+1} and the real-valued function on T, $t \mapsto \max\{\| x \| \mid x \in P_t\}$ is integrable.

Next we assume the existence of utility function $\hat{u} : G \times \hat{P} \to R$ and for all t in T we define $h_t : \hat{P} \to R$ by $h_t(\hat{x}) = \hat{u}(t, \hat{x}(t), \hat{x})$. The assumptions equivalent to (a) and (b) needed in the proof of existence of a strategy in equilibrium are: Assumption (B): for all (t, x) in G, $\hat{u}(t, x, \cdot)$ is continuous and quasi-concave on $\hat{P}$. Assumption (C): For all $\hat{x}$ in $\hat{P}$, $\hat{u}(\cdot, \cdot, \hat{x})$ is Borel-measurable on G. We define, of course, $\hat{x}$ to be in equilibrium if, a.e., $h_t(\hat{x}) \geqslant \hat{u}(t, x, \hat{x})$ for all x in P_t. The proof of existence in this generalized model follows that of Theorem 1. (The proofs of claims 2 and 3 are more complicated in this case; Aumann's[1] Theorem 2 is needed in the proof of claim 2.)

We also mention that in this model under the additional assumption of Theorem 2 there is a T-strategy $\hat{x}$ in equilibrium, s.t.a.e., $\hat{x}(t)$ is an extreme point of P_t. Almost the same proof applies.

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